fundamental result for the classification of X-rays with ordinary radiation, at that time doubtful.

Barkla's other kind of secondary emission is characteristic of the secondary radiator, and is accompanied by selective absorption of the primary rays. He showed that each chemical element emitted more than one definite kind of secondary fluorescent radiation. Concentrating attention on, say, the less penetrating kind, it was found to vary in quality by definite steps with the atomic weight of the secondary radiator.

These results have been confirmed and extended by others since the discovery of the diffraction of X-rays by crystals. This, of course, in no way detracts from the merit of the original discoveries made without this powerful assistance.

The Zeros of Bessel Functions.

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1. Although many results are known concerning the zeros of Bessel functions,* the greater number of these results are of practical importance only in the case of functions of comparatively low order.

For example, McMahon has given a formula† for calculating the zeros of the Bessel function $J_n(x)$, namely that, if $k_1, k_2, k_3, \ldots$, are the positive zeros arranged in ascending order of magnitude, then

$$k_s = \beta - \frac{4n^2-1}{8\beta} - \frac{4(4n^2-1)(28n^2-31)}{3\,.\,(8\beta)^3} - \ldots,$$

where $$\beta = \tfrac{1}{4}\pi(2n+4s-1).$$

Now, this formula is derived from Hankel's well-known asymptotic expansion of $J_n(x)$, and the expansion is available for numerical calculations only when $8x$ is large compared with $4n^2$; and so McMahon's formula fails to fulfil its purpose in the case of the smaller zeros of functions of high order, which are frequently of more importance from the physical point of view than the larger zeros of these functions.

It is, however, possible to obtain a number of quite general existence theorems and descriptive properties concerning Bessel functions of any order; and these results, unlike results obtained by the more classical methods, are of peculiar importance in connection with functions of high order. The theorems in question are derived by a consideration of certain contour integrals which arose* in Debye's investigations of asymptotic expansions of Bessel functions by the method of steepest descents.

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* See Gray and Mathews, 'Bessel Functions,' pp. 34–50, 241–244; Whittaker and Watson, 'Modern Analysis,' Ch. XVII.

† 'Annals of Math.,' vol. 9, pp. 23–30 (1894).

2. As it will be necessary to introduce Bessel functions of the second kind, for which various notations are employed, it is convenient to state here the relations connecting the several functions of the second kind with the fundamental functions $J_{\pm n}(x)$. When n is an integer, the limiting forms of the relations are to be taken.

Hankel's function $Y_n(x)$ is defined† by the equation

$$Y_n(x) = 2\pi e^{n\pi i}\{\cos n\pi \,.\, J_n(x) - J_{-n}(x)\} \operatorname{cosec} 2n\pi,$$

and the functions $\Upsilon_n(x)$, $G_n(x)$ are connected with it by the relations‡

$$\pi \Upsilon_n(x) = -2G_n(x) = e^{-n\pi i}\cos n\pi \,.\, Y_n(x).$$

The two Hankel-Nielsen functions $H_n^{(1)}(x)$, $H_n^{(2)}(x)$, which form a fundamental pair of solutions of Bessel's equation, are defined§ by the equations

$$H_n^{(1)}(x) = J_n(x) + i\Upsilon_n(x), \qquad H_n^{(2)}(x) = J_n(x) - i\Upsilon_n(x);$$

so that

$$J_n(x) = \tfrac{1}{2}\{H_n^{(1)}(x) + H_n^{(2)}(x)\},$$

$$J_{-n}(x) = \tfrac{1}{2}\{e^{n\pi i}H_n^{(1)}(x) + e^{-n\pi i}H_n^{(2)}(x)\}.$$

We shall only consider positive values of n and x, and as it is known‖ that $J_n(x)$ has no zeros when $0 < x \leq n$, we shall, in general, take $x \geq n$; when this is the case, we shall define β to be $\cos^{-1}(n/x)$, where the principal value (*i.e.* the value between 0 and $\frac{1}{2}\pi$) of the inverse cosine is to be taken; and we shall write x and $n \sec \beta$ indifferently.

It is to be noted that $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ are conjugate complex numbers when n and x are real.

3. We now introduce Debye's integral in a form suitable for our analysis; we take the integral of Sommerfield's type, namely,

$$H_n^{(1)}(x) = \frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i} e^{x\sinh w - nw}\, dw,$$

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* 'Math. Ann.,' vol. 67, pp. 535-558 (1910).

† Hankel, 'Math. Ann.,' vol. 1, p. 472. This must, of course, be distinguished from Neumann's function $Y^{(n)}(x)$, which is $(\log 2 - \gamma)\, J_n(x) + \frac{1}{2}Y_n(x)$.

‡ The function $\Upsilon_n(x)$, which is equal to $Y_n(x)/\pi$ when n is an integer, is employed by Gray and Mathews; extensive tables of $G_n(x)$ are published in 'Brit. Ass. Reports,' 1913-1916.

§ Hankel, 'Math. Ann.,' vol. 1, p. 472; Nielsen, 'Cylinderfunktionen,' p. 17.

‖ 'Lond. Math. Soc. Proc.,' (2), vol. 16, pp. 153, 165.

and, observing that $n \sec\beta \sinh w - nw$, *quâ* function of w, has stationary points at $w = \pm i\beta$, we take as the contour of integration a portion of the curve on which

$$\mathrm{I}(\sec\beta \sinh w - w) = \mathrm{I}(\sec\beta \sinh i\beta - i\beta),$$

in accordance with the principles of the method of steepest descents.

Next define a new complex variable t by the equation $w = t + i\beta$, and we get

$$\mathrm{H}_n^{(1)}(n \sec\beta) = \frac{1}{\pi i} e^{ni(\tan\beta - \beta)} \int_{-\infty - i\beta}^{\infty + \pi i - i\beta} e^{-n\tau}\, dt,$$

where $$-\tau = \sinh t - t + i \tan\beta (\cosh t - 1).$$

To investigate the form of the contour we write $u + iv$ for t, where u, v, are real; regarding u, v, as Cartesian co-ordinates, the equation of the contour is

$$\sin(\beta + v) \cosh u = \sin\beta + v \cos\beta.$$

This curve has a double point at the origin,[*] and the lines $v = -\beta$, $v = \pi - \beta$, are asymptotes; the branch of the curve which passes from $(-\infty, -\beta)$ to the origin and thence to $(\infty, \pi - \beta)$ forms a contour of the requisite type; at any point of it u and v are both positive or both negative, while dv/du is always positive. As t describes the contour, τ is real and decreases from $+\infty$ to 0 (at the origin), and then increases to $+\infty$; the value of τ on the contour is

$$u - \sec\beta \sinh u \cos(v + \beta).$$

4. We can now embark on an investigation of the existence of zeros of $\mathrm{J}_n(x)$, and obtain a rough approximation to their positions.

Since each of the integrals

$$\int_{-\infty}^{\infty} e^{-n\tau}\, du, \qquad \int_{-\beta}^{\pi - \beta} e^{-n\tau} dv,$$

is positive, it is clear that, if we regard β as a variable, and if we *define*

$$\arg \int_{-\infty - i\beta}^{\infty + \pi i - i\beta} e^{-n\tau}\, dt$$

to be a positive acute angle when $\beta = 0$, and to vary continuously with β, it will *always* be a positive acute angle; we shall call this angle[†] χ, and then define the angle ψ by the equation

$$\psi = n(\tan\beta - \beta) + \chi.$$

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[*] This statement, together with those immediately following, is contained in Debye's paper. I have proved further, 'Camb. Phil. Soc. Proc.,' vol. 19, p. 105 (1917), that dv/du cannot exceed $\sqrt{3}$.

[†] From the footnote to §3, it follows that $0 < \chi < \frac{1}{2}\pi$.

It is clear that

$$i\,\mathrm{H}_n^{(1)}(n\sec\beta) = \rho e^{i\psi}$$

where ρ is positive; the precise value of ρ is

$$\left|\frac{1}{\pi}\int_{-\infty-i\beta}^{\infty+\pi i-i\beta} e^{-n\tau}\,dt\right|.$$

It is evident that

$$\mathrm{J}_n(x) = \rho\sin\psi, \qquad \Upsilon_n(x) = -\rho\cos\psi;$$

and so the *only* zeros of $\mathrm{J}_n(x)$ are derived* from the values of $\pi, 2\pi, 3\pi, \ldots,$ of ψ.

It will now be shown that, as x increases, ψ also increases; and so to each of the specified values of ψ corresponds *one and only one positive zero of* $\mathrm{J}_n(x)$.

Since

$$\tan\psi = -\mathrm{J}_n(x)/\Upsilon_n(x),$$

it is clear that

$$\frac{d\psi}{dx} = \frac{\mathrm{J}_n(x)\,\Upsilon_n'(x) - \mathrm{J}_n'(x)\,\Upsilon_n(x)}{\mathrm{J}_n^2(x) + \Upsilon_n^2(x)} = \frac{2/(\pi x)}{\mathrm{J}_n^2(x) + \Upsilon_n^2(x)},$$

by a well known formula. The last expression is positive, and so ψ is an increasing function of x.

Hence, as x increases while n remains fixed, ψ increases steadily, and so it passes through the value $m\pi$ once only (m being any one of the integers 1, 2, 3, ...), and, when $\psi = m\pi$, β has a value such that $n(\tan\beta - \beta)$ lies between $(m-\frac{1}{3})\,\pi$ and $m\pi$.

For example, taking $n = 1000$, $\beta = \frac{1}{3}\pi$, we get $n\pi^{-1}(\tan\beta-\beta)$ equal to 218·0, and the value of $\psi\pi^{-1}$ with these values of n and β exceeds this by a positive number less than $\frac{1}{3}$. Hence $\mathrm{J}_{1000}(x)$ has 218 zeros between 1000 and 2000.

5. The results just obtained concerning the zeros of $\mathrm{J}_n(x)$ are derived from the theorem that $d\psi/dx$ is not negative. More precise results can be obtained from the deeper theorem that $d\chi/dx$ is not negative; this theorem we now proceed to prove.

It is easy to show that

$$\begin{aligned} d\chi/dx &= d\psi/dx - n\,d(\tan\beta-\beta)/dx \\ &= \frac{2}{\pi x\,\{\mathrm{J}_n^2(x) + \Upsilon_n^2(x)\}} - \frac{\sqrt{(x^2-n^2)}}{x}, \end{aligned}$$

and so the required theorem is equivalent to the theorem that

$$(x^2-n^2)^{\frac{1}{2}}\,\{\mathrm{J}_n^2(x) + \Upsilon_n^2(x)\} \leq 2/\pi.$$

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* Since $\chi > 0$ and $\tan\beta - \beta \geq 0$, it is evident that ψ is never zero; and since $\mathrm{J}_n(-x) = e^{\pm\frac{1}{2}n\pi i}\,\mathrm{J}_n(x)$ it follows that there is a one-one correspondence between the positive and the negative zeros of $\mathrm{J}_n(x)$.

Now, if $x \to \infty$ while n is fixed, the limit of the function on the left is $2/\pi$; and, consequently, if we can prove that $(x^2-n^2)^{\frac{1}{2}}\{\mathrm{J}_n{}^2(x)+\Upsilon_n{}^2(x)\}$ is an increasing function of x (*i.e.*, that its differential coefficient is not negative), we shall obtain the desired result.

The theorem that $(x^2-n^2)^{\frac{1}{2}}\{\mathrm{J}_n{}^2(x)+\Upsilon_n{}^2(x)\}$ is an increasing function of x seems to be of a deeper nature than might have been anticipated; for example, the proof of it is much more elaborate than the proof of the theorem* that

$$\mathrm{J}_n{}^2(x)+\{x\mathrm{J}_n{}'(x)\}^2/(x^2-n^2)$$

is a decreasing function of x.

Take the formula given by Nicholson†

$$\mathrm{J}_n{}^2(x)+\Upsilon_n{}^2(x) = 8\pi^{-2}\int_0^\infty \mathrm{K}_0(2x\sinh\lambda)\cosh 2n\lambda\, d\lambda,$$

whence the derivate of $(x^2-n^2)^{\frac{1}{2}}\{\mathrm{J}_n{}^2(x)+\Upsilon_n{}^2(x)\}$ is a positive multiple of

$$\int_0^\infty \{x\mathrm{K}_0(2x\sinh\lambda)+2(x^2-n^2)\sinh\lambda\,\mathrm{K}_0{}'(2x\sinh\lambda)\}\cosh 2n\lambda\, d\lambda.$$

To prove that this is positive when $x > n$, we transform it by integrating a portion twice by parts, thus

$$\begin{aligned}
&2n^2\int_0^\infty \sinh\lambda\,\mathrm{K}_0{}'(2x\sinh\lambda)\cosh 2n\lambda\, d\lambda \\
&= \Big[n\sinh\lambda\,\mathrm{K}_0{}'(2x\sinh\lambda)\sinh 2n\lambda\Big]_0^\infty - n\int_0^\infty\{\cosh\lambda\,\mathrm{K}_0{}'(2x\sinh\lambda) \\
&\qquad + 2x\sinh\lambda\cosh\lambda\,\mathrm{K}_0{}''(2x\sinh\lambda)\}\sinh 2n\lambda\, d\lambda \\
&= -n\int_0^\infty 2x\sinh\lambda\cosh\lambda\,\mathrm{K}_0(2x\sinh\lambda)\sinh 2n\lambda\, d\lambda \\
&= \Big[-x\sinh\lambda\cosh\lambda\,\mathrm{K}_0(2x\sinh\lambda)\cosh 2n\lambda\Big]_0^\infty \\
&\qquad + \int_0^\infty\{x\cosh 2\lambda\,\mathrm{K}_0(2x\sinh\lambda) \\
&\qquad + 2x^2\sinh\lambda\cosh^2\lambda\,\mathrm{K}_0{}'(2x\sinh\lambda)\}\cosh 2n\lambda\, d\lambda;
\end{aligned}$$

the result has been simplified after the first step by using the differential equation satisfied by K_0.

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[*] 'Lond. Math. Soc. Proc.,' (2), vol. 16, p. 169.

[†] 'Quart. Journ. Math.,' vol. 42, p. 221 (1911); a complete proof of this formula requires an investigation similar to that given below in §8. As usual, $\mathrm{K}_0(\xi)$ is the Bessel function of Basset's type; it is equal to $\int_0^\infty e^{-\xi\cosh\phi}\,d\phi$, and it satisfies the differential equation

$$\xi\mathrm{K}_0{}''(\xi)+\mathrm{K}_0{}'(\xi)-\xi\mathrm{K}_0(\xi) = 0.$$

Consequently we have

$$\tfrac{1}{8}\pi^2(x^2-n^2)^{\frac{1}{2}}\frac{d}{dx}\left[(x^2-n^2)^{\frac{1}{2}}\{\mathrm{J}_n^2(x)+\Upsilon_n^2(x)\}\right]$$

$$=\int_0^\infty\{-2x\sinh^2\lambda\,\mathrm{K}_0(2x\sinh\lambda)-2x^2\sinh^3\lambda\,\mathrm{K}_0'(2x\sinh\lambda)\}\cosh 2n\lambda\,d\lambda$$

$$=\int_0^\infty\Big\{-2x\sinh^2\lambda\,\mathrm{K}_0(2x\sinh\lambda)$$

$$-x\sinh^3\lambda\,\mathrm{sech}\,\lambda\frac{d}{d\lambda}\mathrm{K}_0(2x\sinh\lambda)\Big\}\cosh 2n\lambda\,d\lambda$$

$$=\int_0^\infty x\,\mathrm{K}_0(2x\sinh\lambda)\Big\{-2\sinh^2\lambda\cosh 2n\lambda$$

$$+\frac{d}{d\lambda}(\sinh^3\lambda\,\mathrm{sech}\,\lambda\cosh 2n\lambda)\Big\}d\lambda,$$

on integrating the second portion of the integral by parts.

Now $\mathrm{K}_0(2x\sinh\lambda)$ is positive, and

$$-2\sinh^2\lambda\cosh 2n\lambda+\frac{d}{d\lambda}(\sinh^3\lambda\,\mathrm{sech}\,\lambda\cosh 2n\lambda)$$

$$=\tanh^2\lambda\cosh 2n\lambda+2n\sinh^3\lambda\,\mathrm{sech}\,\lambda\sinh 2n\lambda \geq 0,$$

so that the last integral has an integrand which is not negative, *i.e.*

$$\frac{d}{dx}[(x^2-n^2)^{\frac{1}{2}}\{\mathrm{J}_n^2(x)+\Upsilon_n^2(x)\}]\geq 0.$$

It follows at once that

$$(x^2-n^2)^{\frac{1}{2}}\{\mathrm{J}_n^2(x)+\Upsilon_n^2(x)\}\leq\lim_{x\to\infty}[(x^2-n^2)^{\frac{1}{2}}\{\mathrm{J}_n^2(x)+\Upsilon_n^2(x)\}]=2/\pi,$$

and so, from the formula given at the beginning of the section,

$$d\chi/dx\geq 0.$$

6. It may be observed, in passing, that $\mathrm{K}_0(2x\sinh\lambda)$ is a decreasing function[*] of x, when λ is assigned, if $x\geq 0$ (whether x be greater or less than n); it follows from Nicholson's integral that

$$\mathrm{J}_n^2(x)+\Upsilon_n^2(x)$$

is a decreasing function of x.

Since $\mathrm{J}_n(x)$ is an increasing function of x when $0<x\leq n$,[†] it follows *a fortiori* that $\Upsilon_n^2(x)$ is a decreasing function of x in this range; and since

[*] This is evident from the formula

$$\mathrm{K}_0(\xi)=\int_0^\infty e^{-\xi\cosh\phi}\,d\phi.$$

[†] 'London Math. Soc. Proc.,' (2), vol. 16, p. 165.

$\Upsilon_n(+0) = -\infty$, it is now evident that, when $0 \leq x \leq n$, $-\Upsilon_n(x)$ is a positive decreasing function of x.

7. From the result of § 5 we can estimate bounds for χ when n is given. We have

$$J_n(x) + i\Upsilon_n(x) = i\rho e^{i(n \tan\beta - n\beta + \chi)}$$

and, from Hankel's asymptotic expansion, when x is large,

$$J_n(x) + i\Upsilon_n(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{ix - \frac{1}{2}(n+\frac{1}{2})\pi i}\{1 + O(1/x)\}.$$

On comparing the arguments of these two expressions, we see that, when x is large and n remains fixed,

$$\chi = x - (\tfrac{1}{2}n + \tfrac{1}{4})\pi - \sqrt{(x^2 - n^2)} + n\cos^{-1}(n/x) + \tfrac{1}{2}\pi + O(1/x).$$

Hence
$$\lim_{x\to\infty} \chi = \tfrac{1}{4}\pi.$$

Also, when $x = n$, we have

$$-i\rho e^{i\chi} = J_n(n) + i\Upsilon_n(n),$$

and so, when $x = n$, $\qquad \chi = \tan^{-1}\{-J_n(n)/\Upsilon_n(n)\}$,

the expression on the right being a positive acute angle.

Since $d\chi/dx$ is not negative, it follows that, when $x \geq n$, we have

$$\tan^{-1}\{-J_n(n)/\Upsilon_n(n)\} \leq \chi < \tfrac{1}{4}\pi.$$

To form an idea of the magnitude of the lower bound of χ we write down Debye's formulæ, valid for large values of n, namely,

$$J_n(n) = \frac{1}{2\pi\sqrt{3}}\left\{\frac{\Gamma(\frac{1}{3})}{(\frac{1}{6}n)^{1/3}} - \frac{\Gamma(\frac{2}{3})}{420\,(\frac{1}{6}n)^{5/3}} - \dots\right\}$$
$$-\Upsilon_n(n) = \frac{1}{2\pi}\left\{\frac{\Gamma(\frac{1}{3})}{(\frac{1}{6}n)^{1/3}} + \frac{\Gamma(\frac{2}{3})}{420\,(\frac{1}{6}n)^{5/3}} - \dots\right\},$$

whence it follows that

$$\lim_{n\to\infty}\{-J_n(n)/\Upsilon_n(n)\} = 1/\sqrt{3} = \tan\tfrac{1}{6}\pi.$$

We write
$$\tan^{-1}\{-J_n(n)/\Upsilon_n(n)\} = \gamma_n,$$
so that
$$\lim_{n\to\infty}\gamma_n = \tfrac{1}{6}\pi;$$

and it appears from Debye's formulæ that, when n is sufficiently large, γ_n is an increasing function of n.

The following Table gives the sexagesimal measure (to the nearest half-minute) of the angle whose circular measure is γ_n; it exhibits the closeness of γ_n to its limit, even when n is quite small:—

n.	0.	$\frac{1}{2}$.	1.	$\frac{3}{2}$.	2.	3.
γ_n	0	26° 34′	29° 23½′	29° 38′	29° 45′	29° 51′

n.	4.	6.	8.	10.	12.	∞.
γ_n	29° 54′	29° 56½′	29° 57½′	29° 58′	29° 58½′	30°

A more precise estimate than the estimate of § 4 can now be made for the positions of the zeros of $J_n(n \sec\beta)$, where β is regarded as the variable; for $\psi = m\pi$ when β has some value for which $n(\tan\beta - \beta)$ lies between $(m-\frac{1}{4})\pi$ and $m\pi - \gamma_n$; and the difference between these two expressions (when n is not small) slightly exceeds $\frac{1}{12}\pi$.

8. The Table just given indicates the possibility of the truth of the proposition that $-J_n(n)/\Upsilon_n(n)$ is an increasing function of n for all positive values of n. Of this proposition we shall now give a formal proof.

It may first be pointed out that (with the notation of § 3)

$$J_n(n) + i\Upsilon_n(n) = \frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i} e^{-n\tau}\,dt,$$

where the contour consists of the negative part of the real axis and a curve on the right of the imaginary axis; it follows that

$$J_n(n) = \frac{1}{\pi}\int_0^{\pi} e^{-n\tau}\,dv$$

$$-\Upsilon_n(n) = \frac{1}{\pi}\int_{-\infty}^{0} e^{-n\tau}\,du + \frac{1}{\pi}\int_0^{\infty} e^{-n\tau}\,du,$$

and the presence of the first integral in the expression for $-\Upsilon_n(n)$ seems to render it impossible to apply successfully a method which has been employed* in proving theorems concerning pairs of Bessel functions of the first kind.

It is, however, possible to construct a proof with the aid of Hardy's theory† of divergent integrals. The theorem to be proved is, effectively, that

$$J_n(n)\frac{d}{dn}\Upsilon_n(n) - \Upsilon_n(n)\frac{d}{dn}J_n(n) \geqq 0.$$

Now the expression on the left is equal to

$$\left\{J_n(x)\frac{\partial}{\partial n}\Upsilon_n(x) - \Upsilon_n(x)\frac{\partial}{\partial n}J_n(x)\right\} + \left\{J_n(x)\frac{d}{dx}\Upsilon_n(x) - \Upsilon_n(x)\frac{d}{dx}J_n(x)\right\},$$

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* 'Lond. Math. Soc. Proc.,' (2), vol. 16, pp. 161–165.

† 'Quart. Journ. Math.,' vol. 35, pp. 22–66 (1904); 'Camb. Phil. Soc. Trans.,' vol. 21, pp. 1–48 (1908).

where x has to be put equal to n after the differentiations have been performed.

If we replace the Bessel functions in the first part of this expression by Hankel-Nielsen functions and apply the theorem quoted in § 4 to the second part of the expression, it is evident that we need to prove that

$$\tfrac{1}{4}n\pi i\left[\mathrm{H}_n^{(1)}(x)\frac{\partial}{\partial n}\mathrm{H}_n^{(2)}(x)-\mathrm{H}_n^{(2)}(x)\frac{\partial}{\partial n}\mathrm{H}_n^{(1)}(x)\right]_{x=n}+1=0.$$

Take the formulæ

$$\mathrm{H}_n^{(1)}(x)=\frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i}e^{x\sinh w-nw}\,dw,$$

$$\mathrm{H}_n^{(2)}(x)=-\frac{1}{\pi i}\int_{-\infty}^{\infty-\pi i}e^{x\sinh \mathrm{W}-n\mathrm{W}}\,d\mathrm{W},$$

in which it is permissible to differentiate with regard to n under the sign of integration.

From the manner in which the integrand tends to zero as $|w|\to\infty$ on the contour specified, it is easy to see that we may introduce an exponential factor of Hardy's type, so that we get*

$$\mathrm{H}_n^{(1)}(x)=\operatorname*{Lim}_{\lambda\to+0}\frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i}\exp(-\lambda w^2)\,.\,e^{x\sinh w-nw}\,dw.$$

By Cauchy's theorem the contour may be deformed into the line $\mathrm{I}(w)=\frac{1}{2}\pi$, so long as $\lambda\neq0$; writing $u+\frac{1}{2}\pi i$ for w, where u is real, we see that

$$\mathrm{H}_n^{(1)}(x)=\operatorname*{Lim}_{\lambda\to+0}\frac{1}{\pi i}e^{-\frac{1}{2}n\pi i}\int_{-\infty}^{\infty}\exp\{-\lambda(u+\tfrac{1}{2}\pi i)^2\}\,.\,e^{ix\cosh u-nu}\,du:$$

As $|\exp(ix\cosh u-nu)|$ does not tend to zero as $u\to-\infty$, it is no longer necessarily permissible† to interchange the limiting process with the process of integration.

In Hardy's notation the last result is written

$$\mathrm{H}_n^{(1)}(x)=\frac{1}{\pi i}e^{-\frac{1}{2}n\pi i}\,.\,\mathrm{G}\int_{-\infty}^{\infty}e^{ix\cosh u-nu}\,du,$$

the expression on the right being described as a "generalised integral."

In like manner, we find that

$$\mathrm{H}_n^{(2)}(x)=-\frac{1}{\pi i}e^{\frac{1}{2}n\pi i}\,.\,\mathrm{G}\int_{-\infty}^{\infty}e^{-ix\cosh \mathrm{U}-n\mathrm{U}}\,d\mathrm{U},$$

with an implied exponential factor, $\exp\{-\lambda(\mathrm{U}-\frac{1}{2}\pi i)^2\}$.

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* That is to say, the sign Lim is commutative with the integral sign: this will not be the case with subsequent integrals.

† *Cf.* Bromwich, 'Theory of Infinite Series,' p. 422.

By considering the contour integrals quoted earlier in the section, it is easy to see that the differentiation with regard to n is commutative with the G-integral signs.

We now substitute the G-integrals in the expression

$$\mathrm{H}_n^{(1)}(x)\frac{\partial}{\partial n}\mathrm{H}_n^{(2)}(x)-\mathrm{H}_n^{(2)}(x)\frac{\partial}{\partial n}\mathrm{H}_n^{(1)}(x);$$

and since the necessary conditions concerning convergence are satisfied for any assigned positive value of λ, we may regard the product of two integrals such as

$$\int_{-\infty}^{\infty} e_1(u)f(u)\,du\times\int_{-\infty}^{\infty} e_2(\mathrm{U})\,\mathrm{F}(\mathrm{U})\,d\mathrm{U},$$

(where e_1, e_2, denote the exponential factors) as a *double integral*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e_1(u)\,e_2(\mathrm{U})\,f(u)\,\mathrm{F}(\mathrm{U})\,.\,(d\mathrm{U}\,du).$$

We thus find that

$$\pi^2\left[\mathrm{H}_n^{(1)}(x)\frac{\partial}{\partial n}\mathrm{H}_n^{(2)}(x)-\mathrm{H}_n^{(2)}(x)\frac{\partial}{\partial n}\mathrm{H}_n^{(1)}(x)\right]_{x=n}$$

$$=\mathrm{G}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\pi i-\mathrm{U}+u)\,e^{in(\cosh u-\cosh \mathrm{U})-n(u+\mathrm{U})}\,(d\mathrm{U}\,du),$$

where the factor $\exp\{-\lambda(u+\frac{1}{2}\pi i)^2-\lambda(\mathrm{U}-\frac{1}{2}\pi i)^2\}$ is implied by the symbol G.

We now change the variables by writing

$$\mathrm{U}=\xi-\eta,\qquad u=\xi+\eta;$$

the usual transformation formula holds, in view of the presence of the exponential factor; and the integral becomes

$$2\mathrm{G}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\pi i+2\eta)\,e^{2ni\sinh\xi\sinh\eta-2n\xi}\,(d\xi\,d\eta),$$

with an implied factor

$$\exp\{-2\lambda\xi^2-2\lambda(\eta+\tfrac{1}{2}\pi i)^2\}.$$

In view of the absolute convergence of the integral, we may replace it by the repeated integral in which the integration with regard to η is performed first.

We therefore consider

$$\int_{-\infty}^{\infty}(\pi i+2\eta)\exp\{-2\lambda(\eta+\tfrac{1}{2}\pi i)^2+2ni\sinh\xi\sinh\eta\}\,d\eta.$$

The modulus of this integral cannot exceed

$$\int_{-\infty}^{\infty}(\pi+2|\eta|)\exp(-2\lambda\eta^2+\tfrac{1}{2}\lambda\pi^2)\,d\eta=\{(2\lambda)^{-1/2}\pi^{3/2}+\lambda^{-1}\}\exp(\tfrac{1}{2}\lambda\pi^2).$$

Further, when ξ is positive, we may deform the η-path of integration into the contour $\mathrm{I}(\eta) = \frac{1}{2}\pi i$; on writing $\eta = v + \frac{1}{2}\pi i$, where v is real, the integral becomes

$$2\int_{-\infty}^{\infty} (v+\pi i) \exp\{-2\lambda(v+\pi i)^2 - 2n \sinh\xi \cosh v\}\, dv$$

$$= 2\exp(2\lambda\pi^2) \,.\, \int_{-\infty}^{\infty} (v+\pi i)\, e^{-2n\sinh\xi\cosh v}\, dv$$

$$+ 2\exp(2\lambda\pi^2) \,.\, \int_{-}^{\infty} (v+\pi i)\{1-\exp(-2\lambda v^2 - 4\lambda v\pi i)\}\, e^{-2n\sinh\xi\cosh v}\, dv$$

$$= 4\pi i \exp(2\lambda\pi^2) \,.\, \int_0^{\infty} e^{-2n\sinh\xi\cosh v}\, dv$$

$$+ 4\exp(2\lambda\pi^2) \,.\, \int_0^{\infty} [iv \sin(4\lambda\pi v)\exp(-2\lambda v^2)$$

$$+ \pi i\{1-\cos(4\lambda\pi v)\exp(-2\lambda v^2)\}]\, e^{-2n\sinh\xi\cosh v}\, dv,$$

on bisecting the ranges of integration and combining the integrals so obtained.

We now approximate to

$$v\sin(4\lambda\pi v)\exp(-2\lambda v^2) + \pi\{1-\cos(4\lambda\pi v)\exp(-2\lambda v^2)\}.$$

The modulus of this expression does not exceed

$$|v\sin(4\lambda\pi v)\exp(-2\lambda v^2) + \pi\{1-\cos(4\lambda\pi v)\}$$

$$+ \pi|\cos(4\lambda\pi v)|\{1-\exp(-2\lambda v^2)\}$$

$$\leq 4\lambda\pi v^2 + 2\pi\sin^2(2\lambda\pi v) + \pi \,.\, 2\lambda v^2$$

$$\leq v^2(6\lambda\pi + 8\lambda^2\pi^3).$$

Hence, when $\xi > 0$, we have (for some value of $|\theta|$ not exceeding 1)

$$\int_{-\infty}^{\infty} (\pi i + 2\eta)\exp\{-2\lambda(\eta+\tfrac{1}{2}\pi i)^2 + 2ni\sinh\xi\sinh\eta\}\, d\eta$$

$$= 4\pi i\exp(2\lambda\pi^2) \,.\, \int_0^{\infty} e^{-2n\sinh\xi\cosh v}\, dv$$

$$+ 2i\theta(6\lambda\pi + 8\lambda^2\pi^3)\exp(2\lambda\pi^2) \,.\, \int_{-\infty}^{\infty} v^2 e^{-2n\sinh\xi\cosh v}\, dv$$

$$= 4\pi i\exp(2\lambda\pi^2)\,\mathrm{K}_0(2n\sinh\xi)$$

$$+ 2i\theta \,.\, (6\lambda\pi + 8\lambda^2\pi^3)\exp(2\lambda\pi^2)\left[\frac{\partial^2}{\partial m^2}\int_{-\infty}^{\infty} e^{-mv-2n\sinh\xi\cosh v}\, dv\right]_{m=0}.$$

The last integral is equal to $2\sec(m\pi)\mathrm{K}_m(2n\sinh\xi)$, where K_m denotes the Bessel function of Basset's type expressible in terms of "the Bessel functions with imaginary argument" by the formula

$$\mathrm{K}_m = \tfrac{1}{2}\pi\cot(m\pi) \,.\, (\mathrm{I}_{-m} - \mathrm{I}_m)$$

We shall write

$$\left[\frac{\partial^2}{\partial m^2}\{\sec(m\pi)\,.\,\mathrm{K}_m(2n\sinh\xi)\}\right]_{m=0}=\mathrm{F}_0(2n\sinh\xi).$$

By using contour integrals of Barnes' type* involving Gamma-functions, it is easy to see that, when ζ is small,

$$\mathrm{F}_0(\zeta)=\mathrm{O}\{(\log\zeta)^3\},$$

while, when ζ is large,

$$\mathrm{F}_0(\zeta)=\mathrm{O}(e^{-\zeta}\zeta^{-3/2}).$$

Thus, when $\xi>0$ we have

$$\int_{-\infty}^{\infty}(\pi i+2\eta)\exp\{-2\lambda(\eta+\tfrac{1}{2}\pi i)^2+2ni\sinh\xi\sinh\eta\}\,d\eta$$

$$=4\pi i\exp(2\lambda\pi^2)\,.\,\mathrm{K}_0(2n\sinh\xi)$$
$$+4i\theta\,.\,(6\lambda\pi+8\lambda^2\pi^3)\exp(2\lambda\pi^2)\,.\,\mathrm{F}_0(2n\sinh\xi),$$

where $|\theta|\leq 1$.

Next consider

$$\int_{-\infty}^{\infty}(\pi i+2\eta)\exp\{-2\lambda(\eta+\tfrac{1}{2}\pi i)^2+2ni\sinh\xi\sinh\eta\}\,d\eta,$$

when $\xi<0$; we may deform the path of integration into the contour $\mathrm{I}(\eta)=-\frac{1}{2}\pi i$; on writing $v-\frac{1}{2}\pi i$ for η, where v is real, the integral becomes

$$\int_{-\infty}^{\infty}2v\exp\{-2\lambda v^2+2n\sinh\xi\cosh v\}\,dv=0,$$

since the integral is an odd function of v.

We now return to the repeated integral

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}(\pi i+2\eta)\exp\{-2\lambda(\eta+\tfrac{1}{2}\pi i)^2+2ni\sinh\xi\sinh\eta\}\,d\eta\right]$$
$$\times\exp(-2\lambda\xi^2-2n\xi)\,d\xi.$$

On dividing the ξ-range of integration into three portions, $(-\infty,-\epsilon)$, $(-\epsilon,\epsilon)$, (ϵ,∞), and substituting for the η-integral in each portion we see that the repeated integral is equal to

$$\int_{-\epsilon}^{\epsilon}\theta_1(2\lambda^{-1/2}\pi^{3/2}+\lambda^{-1})\exp(\tfrac{1}{2}\lambda\pi^2)\exp(-2\lambda\xi^3-2n\xi)\,d\xi$$

$$+\int_{\epsilon}^{\infty}\Big[4\pi i\exp(2\lambda\pi^2)\,.\,\mathrm{K}_0(2n\sinh\xi)$$
$$+4i\theta\,.\,(6\lambda\pi+8\lambda^2\pi^3)\exp(2\lambda\pi^2)\,\mathrm{F}_0)(2n\sinh\xi)\Big]\exp(-2\lambda\xi^2-2n\xi)\,d\xi,$$

where $|\theta_1|<1$, and ϵ may be given *any* positive value.

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* 'Camb. Phil. Soc. Trans.,' vol. 20, pp. 253–279 (1907).

The first integral is equal to

$$2\epsilon\theta_2(2\lambda^{-1/2}\pi^{3/2}+\lambda^{-1})\exp(\tfrac{1}{2}\lambda\pi^2)\exp 2n\epsilon,$$

where $|\theta_2|<1$.

Now make $\epsilon\rightarrow 0$; we find that the repeated integral is equal to

$$4\pi i\exp(2\lambda\pi^2)\,.\int_0^\infty \mathrm{K}_0(2n\sinh\xi)\exp(-2\lambda\xi^2-2n\xi)\,d\xi$$
$$+4i(6\lambda\pi+8\lambda^2\pi^3)\int_0^\infty \theta\,\mathrm{F}_0(2n\sinh\xi)\exp(-2\lambda\xi^2-2n\xi)\,d\xi.$$

The second integral is convergent and its modulus does not exceed

$$\int_0^\infty \mathrm{F}_0(2n\sinh\xi)\,e^{-2n\xi}\,d\xi;$$

this is finite and independent of λ.

Further, in the first of the two integrals, the operation of proceeding to the limit (when $\lambda\rightarrow 0$) is commutative with the integral sign, since K_0 tends exponentially to zero as $n\rightarrow\infty$; and therefore we have proved that

$$2\mathrm{G}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\pi i+2\eta)\,e^{2ni\sinh\xi\sinh\eta-2n\xi}\,(d\xi\,d\eta)=8\pi i\int_0^\infty e^{-2n\xi}\,\mathrm{K}_0(2n\sinh\xi)\,d\xi.$$

Hence we have

$$\tfrac{1}{4}n\pi i\left[\mathrm{H}_n^{(1)}(x)\frac{\partial}{\partial n}\mathrm{H}_n^{(2)}(x)-\mathrm{H}_n^{(2)}(x)\frac{\partial}{\partial n}\mathrm{H}_n^{(1)}(x)\right]_{x=n}+1$$
$$=1-2n\int_0^\infty e^{-2n\xi}\,\mathrm{K}_0(2n\sinh\xi)\,d\xi$$
$$=1-\int_0^\infty e^{-\mu}\,\mathrm{K}_0\left(2n\sinh\frac{\mu}{2n}\right)d\mu.$$

But $2n\sinh(\frac{1}{2}\mu/n)$ is a decreasing function of n when μ is assigned; and K_0 is a positive decreasing function of its argument; and so

$$\int_0^\infty e^{-\mu}\,\mathrm{K}_0\left(2n\sinh\frac{\mu}{2n}\right)d\mu$$

is an increasing function of n, and therefore

$$\tfrac{1}{4}n\pi i\left[\mathrm{H}_n^{(1)}(x)\frac{\partial}{\partial n}\mathrm{H}_n^{(2)}(x)-\mathrm{H}_n^{(2)}(x)\frac{\partial}{\partial n}\mathrm{H}_n^{(1)}(x)\right]_{x=n}+1$$

is a decreasing function of n.

Now the last expression is equal to

$$\tfrac{1}{4}n\pi i\left[\mathrm{H}_n^{(1)}(n)\frac{d}{dn}\mathrm{H}_n^{(2)}(n)-\mathrm{H}_n^{(2)}(n)\frac{d}{dn}\mathrm{H}_n^{(1)}(n)\right],$$

and so this expression is greater than its limit when $n\rightarrow\infty$

But, by Debye's formulæ, when n is large,

$$H_n^{(1)}(n) \sim -\frac{2}{3\pi}\Big[6^{1/3}e^{2\pi i/3}\sin\left(\frac{1}{3}\pi\right)\Gamma\left(\frac{1}{3}\right)n^{-1/3} + \frac{1}{280}\cdot 6^{5/3}e^{10\pi i/3}\sin\left(\frac{5}{3}\pi\right)\Gamma\left(\frac{5}{3}\right)n^{-5/3}+\dots\Big],$$

while $H_n^{(2)}(n)$ is the conjugate of this complex number: and so, since term-by-term differentiations are permissible, we have

$$\tfrac{1}{4}n\pi i\Big[H_n^{(1)}(n)\frac{d}{dn}H_n^{(2)}(n)-H_n^{(2)}(n)\frac{d}{dn}H_n^{(1)}(n)\Big] \sim 1/(105n^2)\to 0$$

as $n\to\infty$.

Hence, for finite values of n,

$$\tfrac{1}{4}n\pi i\Big[H_n^{(1)}(x)\frac{\partial}{\partial n}H_n^{(2)}(x)-H_n^{(2)}(x)\frac{\partial}{\partial n}H_n^{(1)}(x)\Big]_{x=n}+1>0,$$

and we have already seen that this is the condition that $-J_n(n)/Y_n(n)$ should be an increasing function of n.

We have therefore proved the property of the Bessel function which was suggested by the Table given in § 7.

9. So far we have been concerned with what might be regarded as theoretical properties of the zeros of $J_n(x)$; as regards their actual calculation when a rough estimate has been obtained* for the value of the particular zero required, if β is not small, Debye's asymptotic expansion is available; the first two terms of this expansion are given by the formula

$$J_n(n\sec\beta)\sim\left(\frac{2}{n\pi\tan\beta}\right)^{\frac{1}{2}}\Big[\cos\left\{n\tan\beta-\beta)-\frac{1}{4}\pi\right\} + n^{-1}\left(\frac{1}{8}+\frac{5}{24}\cot^2\beta\right)\cot\beta\cos\left\{n(\tan\beta-\beta)-\frac{3}{4}\pi\right\}+\dots\Big],$$

When β is small, an approximation which I have recently communicated to the Cambridge Philosophical Society may be used, namely that, if $\beta\leq\frac{1}{4}\pi$,

$$J_n(n\sec\beta)\sim\tfrac{2}{3}\tan\beta\cos\{n(\tan\beta-\tfrac{1}{3}\tan^3\beta-\beta)-\tfrac{1}{3}\pi\}\,.\,J_{-1/3}(\tfrac{1}{3}n\tan^3\beta) + \tfrac{2}{3}\tan\beta\cos\{n(\tan\beta-\tfrac{1}{3}\tan^3\beta-\beta)+\tfrac{1}{3}\pi\}\,.\,J_{1/3}(\tfrac{1}{3}n\tan^3\beta)$$

with an error less than $24/n$, which is of lower order of magnitude than the approximation (when n is large), except near the zeros of the function.

To apply this formula it is necessary to have tables of the functions $J_{\pm\frac{1}{3}}$: some Tables have been given by Dinnik† of the functions obtained by omitting the factors $\Gamma(1\pm\frac{1}{3})$. These are four-figure Tables with intervals 0·2; on account of the oscillation of the functions (with period roughly equal to 2π) interpolation is impracticable. I have therefore constructed the following two Tables, involving not only the Bessel functions but also the associated functions $U_m(x)$, $V_m(x)$, defined by the equations:

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* By the result obtained in § 7 or otherwise.

† 'Archiv der Math. und Phys.,' (3), vol. 18, pp. 337–338 (1911). Tables of these functions are of importance, as the functions occur in various approximate formulæ and also in the theory of elasticity.

$$J_m(x) = U_m(x)\cos(x-\tfrac{1}{2}m\pi-\tfrac{1}{4}\pi)-V_m(x)\sin(x-\tfrac{1}{2}m\pi-\tfrac{1}{4}\pi),$$

$$J_{-m}(x) = U_m(x)\cos(x+\tfrac{1}{2}m\pi-\tfrac{1}{4}\pi)-V_m(x)\sin(x+\tfrac{1}{2}m\pi-\tfrac{1}{4}\pi).$$

When x is large, $U_m(x)$ and $V_m(x)$ have the well-known asymptotic expansions

$$U_m(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left\{1-\frac{(4m^2-1^2)(4m^2-3^2)}{2\,!\,(8x)^2}+\ldots\right\},$$

$$V_m(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left\{\frac{4m^2-1^2}{1\,!\,(8x)}-\ldots\right\},$$

x.	$J_{\frac{1}{3}}(x)$.	$J_{-\frac{1}{3}}(x)$.	$U_{\frac{1}{3}}(x)$.	$-V_{\frac{1}{3}}(x)$.	$-V_{\frac{1}{3}}(x)/U_{\frac{1}{3}}(x)$.
0·00	0·0000	∞	∞	∞	0·2679
0·05	0·3270	2·5232	2·6937	0·5246	0·1948
0·10	0·4118	1·9971	2·0801	0·3469	0·1668
0·15	0·4703	1·7364	1·7768	0·2629	0·1480
0·20	0·5159	1·5672	1·5833	0·2117	0·1337
0·25	0·5534	1·4425	1·4447	0·1768	0·1224
0·30	0·5850	1·3440	1·3392	0·1517	0·1133
0·35	0·6120	1·2602	1·2535	0·1319	0·1053
0·40	0·6354	1·1879	1·1832	0·1163	0·0983
0·45	0·6555	1·1224	1·1238	0·1037	0·0923
0·50	0·6728	1·0644	1·0727	0·0934	0·0870
0·55	0·6876	1·0097	1·0280	0·0847	0·0823
0·60	0·7000	0·9582	0·9886	0·0773	0·0782
0·65	0·7103	0·9093	0·9534	0·0709	0·0744
0·70	0·7186	0·8623	0·9217	0·0654	0·0709
0·75	0·7250	0·8170	0·8930	0·0604	0·0677
0·80	0·7295	0·7731	0·8668	0·0562	0·0649
0·85	0·7323	0·7321	0·8427	0·0525	0·0622
0·90	0·7334	0·6883	0·8206	0·0491	0·0598
0·95	0·7329	0·6472	0·8001	0·0460	0·0575
1·00	0·7309	0·6069	0·7811	0·0433	0·0554
1·05	0·7275	0·5672	0·7634	0·0408	0·0535
1·10	0·7225	0·5281	0·7468	0·0386	0·0517
1·15	0·7162	0·4896	0·7312	0·0365	0·0499
1·20	0·7085	0·4516	0·7165	0·0346	0·0484
1·25	0·6995	0·4141	0·7027	0·0329	0·0469
1·30	0·6893	0·3771	0·6896	0·0314	0·0455
1·35	0·6780	0·3408	0·6773	0·0299	0·0441
1·40	0·6654	0·3049	0·6656	0·0285	0·0428
1·45	0·6518	0·2696	0·6545	0·0272	0·0416
1·50	0·6372	0·2349	0·6439	0·0261	0·0405
1·55	0·6215	0·2007	0·6338	0·0250	0·0394
1·60	0·6048	0·1672	0·6242	0·0240	0·0384
1·65	0·5873	0·1343	0·6150	0·0230	0·0374
1·70	0·5688	0·1021	0·6062	0·0221	0·0365
1·75	0·5496	0·0706	0·5977	0·0213	0·0357
1·80	0·5296	0·0397	0·5895	0·0205	0·0348
1·85	0·5088	+0·0096	0·5817	0·0198	0·0340
1·90	0·4874	−0·0196	0·5742	0·0191	0·0332
1·95	0·4654	−0·0482	0·5670	0·0184	0·0325
2·00	0·4429	−0·0758	0·5600	0·0178	0·0318

x.	$J_{\frac{1}{3}}(x)$.	$J_{-\frac{1}{3}}(x)$.	$U_{\frac{1}{3}}(x)$.	$-V_{\frac{1}{3}}(x)$.	$-V_{\frac{1}{3}}(x)/U_{\frac{1}{3}}(x)$.
2·0	0·4429	−0·0758	0·5600	0·0178	0·0318
2·2	0·3483	−0·1774	0·5346	0·0157	0·0293
2·4	0·2488	−0·2636	0·5123	0·0139	0·0272
2·6	0·1480	−0·3330	0·4926	0·0125	0·0253
2·8	+0·0490	−0·3847	0·4749	0·0112	0·0237
3·0	−0·0450	−0·4183	0·4590	0·0102	0·0222
3·2	−0·1311	−0·4336	0·4446	0·0093	0·0210
3·4	−0·2071	−0·4314	0·4314	0·0086	0·0198
3·6	−0·2707	−0·4129	0·4194	0·0079	0·0188
3·8	−0·3205	−0·3795	0·4083	0·0073	0·0179
4·0	−0·3554	−0·3331	0·3981	0·0068	0·0170
4·2	−0·3749	−0·2761	0·3886	0·0063	0·0161
4·4	−0·3790	−0·2111	0·3797	0·0058	0·0153
4·6	−0·3680	−0·1408	0·3714	0·0054	0·0147
4·8	−0·3434	−0·0681	0·3637	0·0051	0·0141
5·0	−0·3064	+0·0044	0·3563	0·0048	0·0136
5·2	−0·2589	0·0738	0·3494	0·0046	0·0131
5·4	−0·2031	0·1378	0·3429	0·0044	0·0127
5·6	−0·1415	0·1939	0·3368	0·0041	0·0123
5·8	−0·0765	0·2406	0·3309	0·0039	0·0119
6·0	−0·0106	0·2763	0·3253	0·0037	0·0114
6·2	+0·0534	0·3001	0·3201	0·0036	0·0111
6·4	0·1133	0·3114	0·3151	0·0034	0·0107
6·6	0·1670	0·3100	0·3103	0·0032	0·0104
6·8	0·2126	0·2967	0·3058	0·0031	0·0100
7·0	0·2484	0·2720	0·3013	0·0029	0·0098
7·2	0·2735	0·2374	0·2972	0·0029	0·0096
7·4	0·2872	0·1944	0·2932	0·0027	0·0093
7·6	0·2893	0·1450	0·2893	0·0026	0·0091
7·8	0·2800	0·0911	0·2856	0·0025	0·0089
8·0	0·2598	0·0350	0·2820	0·0024	0·0087

given by Hankel; these may be used for calculation with great accuracy if $x > 8$ when $m = \frac{1}{3}$.

The integrals of Schafheitlin

$$U_m(x),\ V_m(x) = \frac{2^{m+1}\,x^m}{\Gamma(m+\frac{1}{2})\,\Gamma(\frac{1}{2})}\int_0^{\frac{1}{2}\pi} e^{-2x\tan\theta}\sin^{m+\frac{1}{2}}\theta\sec^{2m+1}\theta \,\frac{\cos}{\sin}\,(m-\tfrac{1}{2})\,\theta\,d\theta,$$

which are easily derivable from Hankel's contour integrals, show at once that, when $m = \frac{1}{3}$ and x is positive, $U_{1/3}(x)$ is positive and $V_{1/3}(x)$ is negative.

Further, $V_{1/3}(x)/U_{1/3}(x)$ increases algebraically with x, for

$$U_{1/3}(x)\,V_{1/3}'(x) - U_{1/3}'(x)\,V_{1/3}(x)$$
$$= \left\{\frac{2^{4/3}\,x^{1/3}}{\Gamma(\frac{5}{6})\,\Gamma(\frac{1}{2})}\right\}^2 \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} e^{-2x(\tan\theta+\tan\phi)}\,F(\theta,\,\phi)\,d\theta\,d\phi,$$

where

$$F(\theta,\,\phi) = 2\,(\sin\theta\sin\phi)^{-1/6}\,(\cos\theta\cos\phi)^{-5/3}\,(\tan\phi - \tan\theta)\cos\tfrac{1}{6}\theta\sin\tfrac{1}{6}\phi.$$

Since $F(\theta,\phi)+F(\phi,\theta)$ is positive, if we interchange the parameters θ, ϕ, in the double integral, and add the results so obtained, we find at once that

$$U_{1/3}(x)\,V_{1/3}'(x) - U_{1/3}'(x)\,V_{1/3}(x)$$

is positive.

Hence, as x increases from 0 to ∞, it follows that $V_{1/3}(x)/U_{1/3}(x)$ steadily increases from $-\cot\frac{5}{12}\pi$ to 0, the limit of $V_{1/3}(x)/U_{1/3}(x)$ when $x \to 0$ being derived from its expression in terms of $J_{\pm 1/3}(x)$.

Now the approximate expression for $J_n(n \sec\beta)$ may be written in the form

$$J_n(n\sec\beta) \sim 3^{-\frac{1}{2}}\tan\beta\cos\{n(\tan\beta-\beta)-\tfrac{1}{4}\pi\}\,U_{1/3}(\tfrac{1}{3}n\tan^3\beta)$$
$$-3^{-\frac{1}{2}}\tan\beta\sin\{n(\tan\beta-\beta)-\tfrac{1}{4}\pi\}\,V_{1/3}(\tfrac{1}{3}n\tan^3\beta).$$

Hence, when $\beta \leq \frac{1}{4}\pi$, the zeros of $J_n(n\sec\beta)$ are given approximately by the equation

$$\cot\{n(\tan\beta-\beta)-\tfrac{1}{4}\pi\} = V_{1/3}(\tfrac{1}{3}n\tan^3\beta)/U_{1/3}(\tfrac{1}{3}n\tan^3\beta).$$

If we draw the graphs of the functions

$$V_{1/3}(\tfrac{1}{3}n\tan^3\beta)/U_{1/3}(\tfrac{1}{3}n\tan^3\beta), \qquad \cot\{n(\tan\beta-\beta)-\tfrac{1}{4}\pi\},$$

we see that, near the intersection of the graphs the former has a positive slope and the latter a negative slope, and a good estimate of the value of $\frac{1}{3}n\tan^3\beta$ at the intersection can be obtained by taking values of β on each side of the intersection and then using the principle of proportional parts.

The Electrostatic Problem of a Conducting Sphere in a Spherical Cavity.

By Alexander Russell, M.A., D.Sc., M.I.E.E.

(Communicated by Dr. C. Chree, F.R.S. Received October 4, 1917.)

The mathematical problems which arise in connection with spherical electrodes have often been discussed. A *résumé* of them is given in Chap. VIII, Vol. I, of the second edition of the author's treatise on 'Alternating Current Theory.' The problem, however, of a sphere in a spherical cavity has barely been considered. Sir W. Thomson* (Lord Kelvin) gives numerical results which he says were calculated for application to the theory of conducting spheres "in the interior of a hollow insulated and electrified conductor." As a matter of fact, however, the numerical results given apply only to two equal conducting spheres external to one another.

Solutions for the capacity between two spheres have been given for the case when the centre of the sphere is close to the centre of the cavity. These

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* 'Reprint,' p. 96.